Linearly independent pure-state decomposition and quantum state discrimination

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We put the pure-state decomposition mathematical property of a mixed state to a physical test. We begin by characterizing all the possible decompositions of a rank-two mixed state by means of the complex overlap between two involved states. The physical test proposes a scheme of quantum state recognition of one of the two linearly independent states which arise from the decomposition. We find that the two states associated with the balanced pure-state decomposition have the smaller overlap modulus and therefore the smallest probability of being discriminated conclusively, while in the nonconclusive scheme they have the highest probability of having an error. In addition, we design an experimental scheme which allows to discriminate conclusively and optimally two nonorthogonal states prepared with different *a priori* probabilities. Thus, we propose a physical implementation for this *linearly independent pure-state decomposition and state discrimination* test by using twin photons generated in the process of spontaneous parametric down conversion. The information-state is encoded in one photon polarization state whereas the second single-photon is used for heralded detection.

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I. INTRODUCTION

In quantum information and quantum computing the physical unit of information is a microscopic quantum system and the information is encoded in its state [1]. If the system is isolated, all of its properties are described by a pure state. The fundamental property of a pure state is that it can always be expressed as a coherent superposition of linearly independent states which, for instance, gives account of the accurate quantum interference phenomenon [2]. If the system is not isolated then, in general, it is correlated with an uncontrollable quantum system, usually called environment, which introduces decoherence to the state of the system [3]. In this case the effective state of the system can be described by a mixed state which consists of an incoherent superposition of possible states. In addition, the partial knowledge of the state of a system that belongs to a reservoir also is required to be effectively described by a mixed state. Whatever be the cause for a mixed state, it has the fundamental property of having an infinite number of decompositions. There are some properties that do not depend on the considered decomposition, for instance, eigenvalues, eigenstates, observable-average, purity, and entropy. However, nonorthogonal quantum state discrimination protocol makes use of a mixed composition of the possible prepared states averaged on their *a priori* probabilities. Thus, because of the fundamental property of having an infinite number of decompositions one could discriminate not only the prepared states but also another set of nonorthogonal states belonging to another decomposition of the prepared mix. On the other hand, the entanglement average for a bipartite system depends on the considered decomposition. Since the entanglement is understood as the quantification of resources needed to create a given entangled state, a value of the entanglement of formation of a mixed state is then defined to be the average entanglement of the pure-state decomposition, minimized over all possible decompositions [4]. In this form, the mathematical-decomposition property is a nonclassical characteristic [5] and becomes of fundamental physical interest.

When the state belongs to a set of linearly independent (LI) and nonorthogonal states, it can be recognized conclusively with a probability different from zero [6–10]. Another strategy allows a minimum-error tolerance for discriminating the state [11–13]. In the simplest case the state is in a two-dimensional Hilbert space and only two different states are LI. In a Hilbert space with dimension higher than two, the states require additional constraints to be LI [14]. In order to simplify the study of the addressed problem we have considered a rank-two mixed state. In this work we relate the LI pure-state decomposition property of a given mixed state to the unambiguous quantum state discrimination (UQSD) protocol for two nonorthogonal states.

II. MIXED STATE DECOMPOSITION

We consider a quantum system prepared in a mixed state whose spectral decomposition is given by

$$\rho = \lambda_1 |\lambda_1\rangle\langle\lambda_1| + \lambda_2 |\lambda_2\rangle\langle\lambda_2|, \tag{1}$$

being $\{|\lambda_1\rangle, |\lambda_2\rangle\}$ the eigenstates and $\{\lambda_1, \lambda_2\}$ the eigenvalues respectively. We recall that $\lambda_i \in [0,1]$, $\lambda_1+\lambda_2=1$, and $\langle\lambda_1|\lambda_2\rangle = 0$. We shall assume λ_1 or λ_2 to be different from zero. This spectral-decomposition is unique and its purity is completely characterized by the eigenvalues. However, there are infinite possible *pure-state decompositions* for the same mixed state [4]. Here we introduce two nonorthogonal states $|\beta_1\rangle$ and $|\beta_2\rangle$ with inner product

$\langle\beta_1|\beta_2\rangle = \beta$. In terms of these states and their biorthogonal ones [15] the identity can be represented as follows:

$$I = \frac{|\beta_1\rangle - \beta^*|\beta_2\rangle}{1-|\beta|^2}\langle\beta_1| + \frac{|\beta_2\rangle - \beta|\beta_1\rangle}{1-|\beta|^2}\langle\beta_2|. \quad (2)$$

This expression becomes the well known canonical one for $\beta = 0$. Making use of the (2) identity we can find all the decompositions of ρ whose forms are:

$$\rho = p_1|\beta_1\rangle\langle\beta_1| + p_2|\beta_2\rangle\langle\beta_2|. \quad (3)$$

Here p_1 and p_2 play the role of the *a priori* probabilities associated with the $|\beta_1\rangle$ and $|\beta_2\rangle$ states, respectively. After some algebra we obtain

$$p_1 = \frac{\lambda_1\lambda_2}{\lambda_1 + (\lambda_2-\lambda_1)|\gamma|^2}, \quad (4a)$$

$$p_2 = \frac{\lambda_1^2 + (\lambda_2-\lambda_1)|\gamma|^2}{\lambda_1 + (\lambda_2-\lambda_1)|\gamma|^2}, \quad (4b)$$

satisfying $0 \le p_i \le 1$ and $p_1 + p_2 = 1$. We note that p_1, as a function of $|\gamma|$, is monotonically increasing (for $\lambda_1 < \lambda_2$) or decreasing (for $\lambda_1 > \lambda_2$) enclosed by λ_1 and λ_2. The $|\gamma|$-decomposition states become

$$|\beta_1\rangle = \gamma|\lambda_1\rangle + \sqrt{1-|\gamma|^2}|\lambda_2\rangle, \quad (5a)$$

$$|\beta_2\rangle = \frac{\lambda_1\sqrt{1-|\gamma|^2}|\lambda_1\rangle - \lambda_2\gamma^*|\lambda_2\rangle}{\sqrt{\lambda_1^2 + (\lambda_2-\lambda_1)|\gamma|^2}}, \quad (5b)$$

where the $\gamma = |\gamma|e^{i\theta}$ parameter is the component of the $|\lambda_1\rangle$ eigenstate in the $|\beta_1\rangle$ state, in short $\gamma = \langle\lambda_1|\beta_1\rangle$, and the phase θ is the relative phase of the allowed $\{|\beta_i\rangle\}$ states. From the (5) expressions we realize that the inner product between the allowed $\{|\beta_i\rangle\}$-decomposition states is given by

$$\langle\beta_1|\beta_2\rangle = \frac{(\lambda_1-\lambda_2)|\gamma|\sqrt{1-|\gamma|^2}}{\sqrt{\lambda_1^2 + (\lambda_2-\lambda_1)|\gamma|^2}}e^{-i\theta}. \quad (6)$$

We notice that, as is evident, when $\lambda_1 = \lambda_2$ all the possible decompositions are one half of the identity since in this case we get $p_1 = p_2$ as well, and all the possible sets $\{|\beta_i\rangle\}$ are given by the orthogonal states:

$$|\beta_1\rangle = |\gamma|e^{-i\theta}|\lambda_1\rangle + \sqrt{1-|\gamma|^2}|\lambda_2\rangle, \quad (7a)$$

$$|\beta_2\rangle = \sqrt{1-|\gamma|^2}|\lambda_1\rangle - |\gamma|e^{i\theta}|\lambda_2\rangle. \quad (7b)$$

From now on we assume $\lambda_1 \neq \lambda_2$. On the other hand, the (1) spectral-decomposition is recovered for both values $|\gamma| = 1$ and $|\gamma| = 0$.

The modulus of the overlap (6) is a convex function of $|\gamma|$ being zero for $|\gamma| = 0$, 1 and its maximum value is reached for

$$|\gamma| = \sqrt{\lambda_1}. \quad (8)$$

In this case the decomposition corresponds to the *balanced* one since $p_1 = p_2 = 1/2$, and the states become

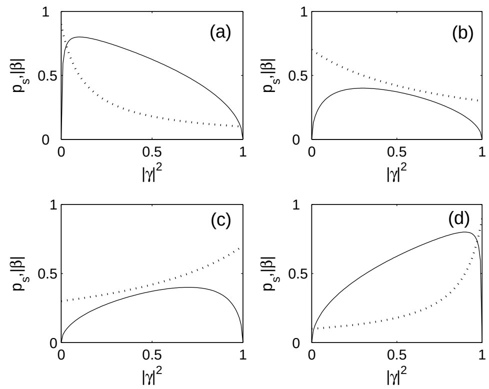

FIG. 1: Probability p_1 (dotted line) and $|\beta| = |\langle\beta_1|\beta_2\rangle|$ (solid line) as functions of $|\gamma|^2$ for different values of λ_1: (a) $\lambda_1 = 0.1$, (b) $\lambda_1 = 0.3$, (c) $\lambda_1 = 0.7$, and (d) $\lambda_1 = 0.9$.

$$|\beta_1\rangle = \sqrt{\lambda_1}e^{-i\theta}|\lambda_1\rangle + \sqrt{\lambda_2}|\lambda_2\rangle, \quad (9a)$$

$$|\beta_2\rangle = \sqrt{\lambda_1}|\lambda_1\rangle - \sqrt{\lambda_2}e^{i\theta}|\lambda_2\rangle, \quad (9b)$$

whose overlap is

$$\langle\beta_1|\beta_2\rangle = (\lambda_1 - \lambda_2)e^{i\theta}. \quad (10)$$

Thus, the *balanced-decomposition* states have the maximal modulus of the $\langle\beta_1|\beta_2\rangle$ overlap. In other words, for that decomposition the two states are as close as possible.

Figure 1 shows the p_1 probability (solid) and the overlap modulus, $|\beta|$, (dashed) as a function of $|\gamma|^2$ for different values of λ_1. We note that p_1 is between λ_1 and λ_2 and $|\beta|$ reaches its maximal value at $|\gamma| = \sqrt{\lambda_1}$.

By implementing the measurement procedures of two observables, each one with eigenstates $\{|\beta_1\rangle, |\overline{\beta}_1\rangle, \langle\beta_1|\overline{\beta}_1\rangle = 0\}$ and $\{|\beta_2\rangle, |\overline{\beta}_2\rangle, \langle\beta_2|\overline{\beta}_2\rangle = 0\}$, we obtain the overlap modulus between the states of the considered decomposition, because of the identity:

$$\langle\beta_1|\rho|\beta_1\rangle + \langle\beta_2|\rho|\beta_2\rangle = 1 + |\langle\beta_1|\beta_2\rangle|^2. \quad (11)$$

Thus, the non-orthogonality of the basis $\{|\beta_1\rangle, |\beta_2\rangle\}$ gives the displacement from 1 in the square modulus of the overlap β.

In this section we have characterized by means of the parameter $|\gamma| = |\langle\beta_1|\lambda_1\rangle|$ all possible *pure-state decompositions* (3) of a given two-rank mixed state (1). The modulus of the overlap between the states of the decomposition goes from zero for the spectral-decomposition up to $|\lambda_1 - \lambda_2|$ for the balanced-decomposition whereas its phase is $-\theta$ when $\lambda_1 > \lambda_2$ or $-\theta + \pi$ when $\lambda_1 < \lambda_2$. In the next section we relate a possible $|\gamma|$-decomposition of a given density operator to the process for unambiguous nonorthogonal quantum state discrimination.

III. QUANTUM STATE DISCRIMINATION

Dieks, Ivanovic, and Peres [7, 17, 18] addressed the fundamental problem of discriminating conclusively and unambiguously between two non-orthogonal states, $|\beta_1\rangle$ and $|\beta_2\rangle$, which are randomly prepared in a quantum system with *a priori* probabilities p_1 and p_2 respectively. The optimal success probability for removing the doubt as to which $|\beta_1\rangle$ or $|\beta_2\rangle$ the system is in, was derived by Jeager and Shimony [19] obtaining the expressions

$$p_s = 1 - 2\sqrt{p_1 p_2}|\beta| \tag{12}$$

when $|\beta| \in [0, \min\{\sqrt{p_1/p_2}, \sqrt{p_2/p_1}\}]$, and

$$p_s = (1-|\beta|^2)\max\{p_1, p_2\} \tag{13}$$

when $|\beta| \in [\min\{\sqrt{p_1/p_2}, \sqrt{p_2/p_1}\}, 1]$. Replacing in these formulas both p_1 and p_2 from Eqs. (4) and $|\beta|$ from Eq. (6) we obtain the optimal probability of success for discriminating unambiguously the two non-orthogonal states of the $|\gamma|-$decomposition of a given ρ mixed state,

$$p_s = \begin{cases} 1 - 2\frac{|\lambda_1-\lambda_2||\gamma|\sqrt{\lambda_1\lambda_2\left(1-|\gamma|^2\right)}}{\lambda_1+(\lambda_2-\lambda_1)|\gamma|^2} & \text{if} \quad 0 \le |\beta| \le \min\left\{\sqrt{\frac{p_1}{p_2}}\ ,\ \sqrt{\frac{p_2}{p_1}}\right\}, \\ \left(1 - \frac{|\lambda_1-\lambda_2|^2|\gamma|^2\left(1-|\gamma|^2\right)}{\lambda_1^2+(\lambda_2-\lambda_1)|\gamma|^2}\right)\frac{\max\left\{\lambda_1\lambda_2\ ,\ \lambda_1^2+(\lambda_2-\lambda_1)|\gamma|^2\right\}}{\lambda_1+(\lambda_2-\lambda_1)|\gamma|^2} & \text{if} \quad \min\left\{\sqrt{\frac{p_1}{p_2}}\ ,\ \sqrt{\frac{p_2}{p_1}}\right\} \le |\beta| \le 1. \end{cases} \tag{14}$$

As we know, for a given λ_1 the optimal probability p_s takes its highest value, 1, for the extreme values $|\gamma| = 0, 1$ which correspond to the spectral-decomposition, whereas it reaches the smallest value just for $|\gamma| = \sqrt{\lambda_1}$ which corresponds to the balanced-decomposition. In other words, the states belonging to the balanced-decomposition have the smallest optimal probability of being unambiguously discriminated. In this case the probability of success becomes $1 - |\lambda_1 - \lambda_2|$.

A non-trivial relation between $|\gamma|$ and λ_1 is obtained from the intervals defined by $|\beta|$ and $\min\{\sqrt{p_1/p_2},\sqrt{p_2/p_1}\}$ in Eq. (14). Figure 2(a) shows the regions of the $(|\gamma|^2,\lambda_1)$ plane where $0 \le |\beta| \le \min\{\sqrt{p_1/p_2},\sqrt{p_2/p_1}\}$ (gray) and where $\min\{\sqrt{p_1/p_2},\sqrt{p_2/p_1}\} \le |\beta| \le 1$ (black $p_1 \le p_2$ and white $p_1 \ge p_2$). It is worth emphasizing that in the gray area of the Fig. 2(a) both states $|\beta_1\rangle$ and $|\beta_2\rangle$ can be unambiguously discriminated whereas in the white and black zones only the state associated with the higher probability p_1 or p_2 is discriminated. Specifically, in the white area only $|\beta_1\rangle$ can be discriminated and in the black one only $|\beta_2\rangle$. In Fig. 2(b) we plot in degradation black-gray-white the optimal success probability (14) as a function of $|\gamma|^2$ and λ_1. In Fig. 3 we show the (14) probability (solid lines) as functions of $|\gamma|^2$ for different values of λ_1. Notice that p_s as a function of $|\gamma|^2$ is antisymmetric with respect to $\lambda_1 = 1/2$ and the minimal values are just at $|\gamma| = \sqrt{\lambda_1}$.

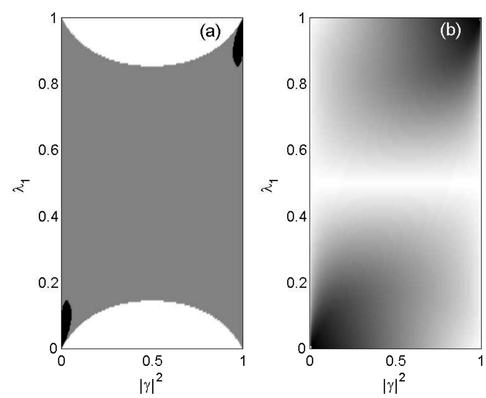

FIG. 2: (a) regions of the plane $(|\gamma|^2,\lambda_1)$ where $0 \le |\beta| \le \min\{\sqrt{p_1/p_2},\sqrt{p_2/p_1}\}$ (gray) and where $\min\{\sqrt{p_1/p_2},\sqrt{p_2/p_1}\} \le |\beta| \le 1$ (black and white), (b) the p_s optimal probability for unambiguous discriminating of the $\{|\beta_i\rangle\}$ states as a function of $|\gamma|^2$ and λ_1. Black color stands for $p_s = 0$, white color means $p_s = 1$, and the gray degradation goes linearly from 0 to 1.

On the other hand, the two states of the $|\gamma|$-decomposition could be recognized tolerating an error. In this case the strategy of discriminating them with minimum-error leads to the Helstrom limit [11]. In other words, the probability of having the smallest error in the identification of the state is

$$p_e = \frac{1}{2}\left(1 - \sqrt{1 - 4p_1p_2|\langle\beta_1|\beta_2\rangle|^2}\right). \tag{15}$$

Replacing in this expression p_1 and p_2 from Eqs. (4) and $|\langle\beta_1|\beta_2\rangle|$ from Eq. 6 we obtain the probability of discriminating with minimum error the $|\gamma|$-decomposition states,

$$p_e = \frac{1}{2}\left(1 - \sqrt{1 - 4\frac{\lambda_1\lambda_2|\lambda_1-\lambda_2|^2|\gamma|^2(1-|\gamma|^2)}{[\lambda_1+(\lambda_2-\lambda_1)|\gamma|^2]^2}}\right). \tag{16}$$

This probability reaches its smallest value, 0, in the extreme values $|\gamma| = 0$ and $|\gamma| = 1$ which correspond to the spectral-decomposition whereas it has the highest value just for $|\gamma| = \sqrt{\lambda_1}$ which corresponds to the balanced-decomposition. Thus, the states belonging to the balanced-decomposition have the highest probability of discriminating them with minimum-error and this is $\frac{1}{2}\left(1 - \sqrt{1 - |\lambda_1 - \lambda_2|^2}\right)$. In Fig. 3 we show the (16) probability (dotted lines) as function of $|\gamma|^2$ for different values of λ_1. We can see that it has its maximal values for the states of the balanced-decomposition ($|\gamma| = \sqrt{\lambda_1}$) and has the minimum values, 0, for the states of the spectral one ($|\gamma| = 0, 1$).

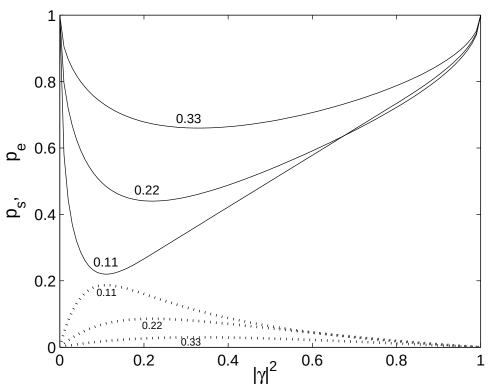

FIG. 3: Optimal success probability p_s (solid lines) and optimal probability of minimum-error p_e (dotted lines) as functions of $|\gamma|^2$ for different values of λ_1, to say: 0.11, 0.22, and 0.33. The respective values of λ_1 are indicated for each of the curves.

IV. EXPERIMENTAL SCHEME FOR OPTIMAL UQSD

For the unambiguous states discrimination protocol we propose a modified version of the experimental setup sketched in Ref. [9], see Fig. 4. We denote by $|h\rangle$ the horizontal and by $|v\rangle$ the vertical polarization photon states. For increasing the Hilbert space we consider an ancillary system which consists of a set of four orthogonal effective distinguishable propagation paths denoted by the states $|1\rangle_p$, $|2\rangle_p$, $|2'\rangle_p$, and $|2''\rangle_p$ as shows Fig. 4.

We assume that the two nonorthogonal possible states $|\beta_i\rangle$, each one having *a priori* probability p_i, enter asymmetrically with respect to the horizontal polarization photon state $|h\rangle$, specifically,

$$|\beta_1\rangle = \cos x|h\rangle + \sin x|v\rangle, \tag{17a}$$
$$|\beta_2\rangle = \cos(\alpha - x)|h\rangle - \sin(\alpha - x)|v\rangle. \tag{17b}$$

A PBS transmits the horizontal polarization and reflects the vertical one introducing in addition a phase of $\pi/2$.

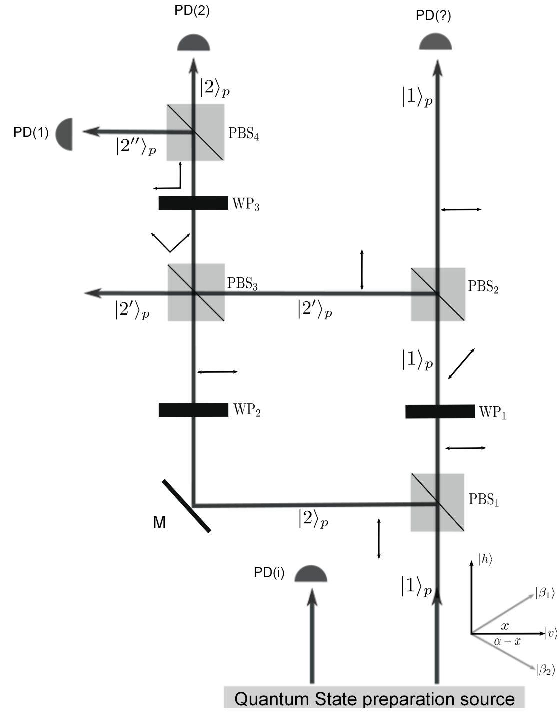

FIG. 4: Sketch of the experimental setup for conclusively discriminating nonorthogonal quantum states. We have denoted by WP the wave plate, by PBS the polarizing beam splitter, by M the mirror, and by PD the single-photon photodiode detectors.

Thus, after the photon passes through the PBS1 the $|\beta_i\rangle|1\rangle_p$ states are transformed as follows:

$$\begin{aligned}
|\beta_1\rangle|1\rangle_p &\rightarrow \cos x|h\rangle|1\rangle_p + i\sin x|v\rangle|2\rangle_p, \\
|\beta_2\rangle|1\rangle_p &\rightarrow \cos(\alpha - x)|h\rangle|1\rangle_p - i\sin(\alpha - x)|v\rangle|2\rangle_p.
\end{aligned}$$

We consider that the WP1 rotates the photon polarization state $|h\rangle$ an angle ϕ, and the WP2 rotates it φ. Therefore, the states change to

$$\begin{aligned}
|\beta_1\rangle|1\rangle_p &\rightarrow \cos x(\cos\phi|h\rangle + \sin\phi|v\rangle)|1\rangle_p \\
&\quad + i\sin x(\sin\varphi|h\rangle - \cos\varphi|v\rangle)|2\rangle_p, \\
|\beta_2\rangle|1\rangle_p &\rightarrow \cos(\alpha - x)(\cos\phi|h\rangle + \sin\phi|v\rangle)|1\rangle_p \\
&\quad - i\sin(\alpha - x)(\sin\varphi|h\rangle - \cos\varphi|v\rangle)|2\rangle_p.
\end{aligned}$$

The unitary effect of the PBS2 on the previous states is

$$\begin{aligned}
|\beta_1\rangle|1\rangle_p &\rightarrow \cos x(\cos\phi|h\rangle|1\rangle_p + i\sin\phi|v\rangle|2'\rangle_p) \\
&\quad + i\sin x(\sin\varphi|h\rangle - \cos\varphi|v\rangle)|2\rangle_p, \\
|\beta_2\rangle|1\rangle_p &\rightarrow \cos(\alpha - x)(\cos\phi|h\rangle|1\rangle_p + i\sin\phi|v\rangle|2'\rangle_p) \\
&\quad - i\sin(\alpha - x)(\sin\varphi|h\rangle - \cos\varphi|v\rangle)|2\rangle_p.
\end{aligned}$$

Meanwhile the unitary effect of the PBS3 transforms them as follows:

$$\begin{aligned}
|\beta_1\rangle|1\rangle_p &\rightarrow \cos x\cos\phi|h\rangle|1\rangle_p + i\sqrt{q_{s1}}|\eta_1\rangle|2\rangle_p \\
&\quad + \sin x\cos\varphi|v\rangle|2'\rangle_p, \\
|\beta_2\rangle|1\rangle_p &\rightarrow \cos(\alpha - x)\cos\phi|h\rangle|1\rangle_p - i\sqrt{q_{s2}}|\eta_2\rangle|2\rangle_p \\
&\quad - \sin(\alpha - x)\cos\varphi|v\rangle|2'\rangle_p,
\end{aligned}$$

where we have defined the normalized states

$$|\eta_1\rangle = \frac{\sin x \sin\varphi|h\rangle + i\cos x\sin\phi|v\rangle}{\sqrt{q_{s1}}}, \quad (22a)$$

$$|\eta_2\rangle = \frac{\sin(\alpha-x)\sin\varphi|h\rangle - i\cos(\alpha-x)\sin\phi|v\rangle}{\sqrt{q_{s2}}}, \quad (22b)$$

and the probabilities

$$q_{s1} = \cos^2 x\sin^2\phi + \sin^2 x\sin^2\varphi,$$
$$q_{s2} = \cos^2(\alpha-x)\sin^2\phi + \sin^2(\alpha-x)\sin^2\varphi.$$

From Eqs. (22) we realize that conclusive discrimination can be performed if $|\eta_1\rangle$ and $|\eta_2\rangle$ are orthogonal. This requirement is satisfied when

$$\sin^2\phi = \tan x\tan(\alpha-x)\sin^2\varphi. \quad (24)$$

It is important to point out that the initial angles α and x are restricted in such a way that the right side of Eq. (24) has to be higher than or equal to 0 and lower than or equal to 1. Specifically, it is satisfied for all ϕ and φ when $0 \le x \le \alpha$. Therefore, by considering satisfied the conditions (24) and $0 \le x \le \alpha$, the conclusive discrimination of the nonorthogonal $|\beta_i\rangle$ states becomes just the discrimination between the two orthogonal polarizations $|\eta_1\rangle$ and $|\eta_2\rangle$ of the single photon in the path $|2\rangle_p$. Replacing the expression (24) in the probabilities q_{si} we find, as a function of x, the probability $p_s(x)$ of successfully discriminating the $|\beta_i\rangle$ states, this is

$$\begin{aligned} p_s(x) &= p_1 q_{s1} + p_2 q_{s2}, \\ &= \left[p_1\frac{\sin x}{\cos(\alpha-x)} + p_2\frac{\sin(\alpha-x)}{\cos x}\right]\sin\alpha\sin^2\varphi. \end{aligned} \quad (25)$$

The first term $p_1 q_{s1}$ corresponds to the probability of discriminating the $|\beta_1\rangle$ state and the second term for $|\beta_2\rangle$. For $x = 0$ ($x = \alpha$) there is no probability of discriminating $|\beta_1\rangle$ ($|\beta_2\rangle$). For other values of x both states can be discriminated with probabilities different from zero. We can also note that when the initial states $|\beta_i\rangle$ are prepared symmetrically ($x = \alpha/2$) with respect to the horizontal polarization, the probability (25) does not depend on the a priori probabilities p_1 and p_2. Therefore, the asymmetry is necessary for the optimization. Figure 5 shows $p_s(x)$ as a function of x for different values of α and p_1. Note that, depending on the values of α and p_1, the function $p_s(x)$ has its maximal value inside the interval or at one of the extremes values of x, to say: $x = 0$ if $p_1 < p_2$ or $x = \alpha$ if $p_1 > p_2$. From Eq. (25) one analytically finds that the optimal value of the total probability of success, $p_s(x)$, is in x such that

$$\cos x = \frac{\sqrt{p_2}\sin\alpha}{\sqrt{1-2\sqrt{p_1p_2}\cos\alpha}},$$

and the maximal one becomes

$$p_{s,\max} = \begin{cases} \left(1-2\sqrt{p_1p_2}\cos\alpha\right)\sin^2\varphi & \text{if} \quad 0 \le \cos\alpha \le \min\left\{\sqrt{\frac{p_1}{p_2}}, \sqrt{\frac{p_2}{p_1}}\right\}, \\ \left(1-\cos^2\alpha\right)\max\{p_1,p_2\}\sin^2\varphi & \text{if} \quad \min\left\{\sqrt{\frac{p_1}{p_2}}, \sqrt{\frac{p_2}{p_1}}\right\} \le \cos\alpha \le 1, \end{cases} \quad (26)$$

which is just the well known Jeager and Shimony formula (14) for $\varphi = \pm\pi/2$ (here $\cos\alpha = |\langle\beta_1|\beta_2\rangle|$) [19].

In this optimal case, the $|\eta_i\rangle$ states of Eqs. (22) become

$$|\eta_1\rangle = \cos\xi|h\rangle + \sin\xi|v\rangle, \quad (27a)$$
$$|\eta_2\rangle = -\sin\xi|h\rangle + \cos\xi|v\rangle, \quad (27b)$$

where

$$\cos\xi = \frac{\sqrt{p_1-\sqrt{p_1p_2}\cos\alpha}}{\sqrt{1-2\sqrt{p_1p_2}\cos\alpha}} \quad (28)$$

when $0 \le \cos\alpha \le \min\{\sqrt{p_1/p_2}, \sqrt{p_1/p_2}\}$; and

$$|\eta_1\rangle = i|v\rangle, \quad (29a)$$
$$|\eta_2\rangle = |h\rangle, \quad (29b)$$

when $\min\{\sqrt{p_1/p_2}, \sqrt{p_1/p_2}\} \le \cos\alpha \le 1$ and $p_1 < p_2$; or

$$|\eta_1\rangle = |h\rangle, \quad (30a)$$
$$|\eta_2\rangle = -i|v\rangle, \quad (30b)$$

when $\min\{\sqrt{p_1/p_2}, \sqrt{p_1/p_2}\} \le \cos\alpha \le 1$ and $p_1 > p_2$.

The WP3 (see Fig. 4) rotates the orthogonal photon polarized state (27) in such a way that $|\eta_1\rangle \to |v\rangle$ and $|\eta_2\rangle \to |h\rangle$. In this form the PBS4 takes the orthogonal outcome polarization states into the detector PD(1) or PD(2) with optimal probability. When the process is optimized with respect to x and φ ($\varphi = \pm\pi/2$) there is no outcome through the path $|2'\rangle_p$ and so the inconclusive outcome through the path $|1\rangle_p$ is detected with minimal probability $1 - p_{s,\max}$ at the PD(?) photo-detector. On the other hand, if $\min\{\sqrt{p_1/p_2}, \sqrt{p_1/p_2}\} \le \cos\alpha \le 1$,

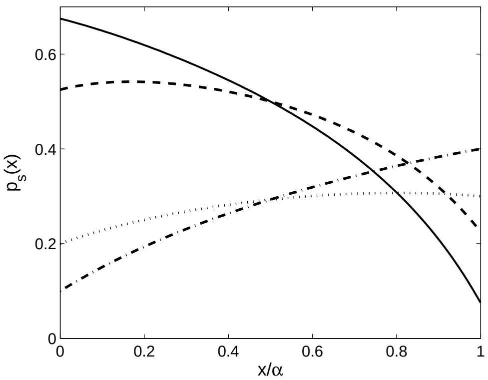

FIG. 5: Success probability $p_s(x)$ as functions of x for different values of α, and p_1, say: $\alpha = \pi/3$ with $p_1 = 0.1$ (solid) and $p_1 = 0.3$ (dashes), $\alpha = \pi/4$ with $p_1 = 0.6$ (dots) and $p_1 = 0.8$ (dash-dot).

the $|\eta_i\rangle$ states coincide with the vertical and the horizonal polarization states as can be seen from Eqs. (29) and (30); therefore in this case the WP3 is not required.

Thus, we have designed a physical scheme for discriminating conclusively and optimally two nonorthogonal states associated with different *a priori* probabilities. Therefore, this designed experimental setup allows one to discriminate desired $|\gamma|$-decomposition states of a two-rank mixed state.

V. SUMMARY

In this work we have presented a physical test for the LI pure-state decomposition property of a rank-two mixed state. We characterized by a complex parameter all the possible LI pure-state decompositions of a mixed state lying in a two-dimensional Hilbert space. The physical test consists of performing a process of recognition of one of the two linearly independent pure-states which arise from a desired decomposition. We find that the two states associated with the balanced pure-state decomposition have the smallest probability of being conclusively discriminated while in the nonconclusive scheme they have the highest probability of having an error. In addition, we designed an experimental scheme which allows one to discriminate conclusively and optimally two nonorthogonal states prepared with different *a priori* probabilities. We have proposed an experimental implementation for this *linearly independent pure-state decomposition* and UQSD test by using a one-photon polarization state generated in the process of spontaneous parametric down conversion (SPDC) where the second single-photon is considered for heralded detection.

For preparing the (1) state the scheme described in Ref. [20] can be implemented. The signal (s) and idler (i) twin photons are generated noncollinearly by SPDC in the normalized state $|\Psi\rangle_{s,i} = \sqrt{\lambda_1}|h\rangle_s|h\rangle_i + \sqrt{\lambda_2}|v\rangle_s|v\rangle_i$. In this form, by ignoring the polarized state of the idler photon we get the state (1) for the signal photon with $|\lambda_1\rangle = |h\rangle$ and $|\lambda_2\rangle = |v\rangle$. The experiment described in Ref. [20] was implemented by using a 351.1 nm single-mode Ar-ion laser pump with a 200 mW and 5-mm-thick BBO crystal, cut for type-II phase matching which allows a higher stability. Our proposed scheme for *linearly independent pure-state decomposition* and *unambiguous quantum state discrimination* could also be implemented with thit setup.

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